

Symbolic powers of planar point configurations II

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Abstract

In [12] we began to study the initial sequences $\alpha(I^{(m)})$, $m = 1, 2, 3 \dots$, of radical ideals I of finite sets of points in the projective plane. In the present note we complete results obtained in [12] by answering a number of questions left open in the previous note and we extend our considerations to the asymptotic setting of Waldschmidt constants. The concept of the Bezout decomposition introduced in Definition 2.4 might be of independent interest.

Keywords symbolic power, fat points, postulation problems, point configurations

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1 Introduction

Symbolic powers of ideals of point configurations have attracted considerable attention in recent years. Apart from traditional paths of research motivated by various problems in Algebraic Geometry, Commutative Algebra and Combinatorics (see e.g. [6], [18], [20]), ideals of planar points have been recently studied in connection with the counterexamples to the $I^{(3)} \subset I^2$ containment (see e.g. [13], [5], [11], [17]) and with the Bounded Negativity Conjecture (see [2]). These recent directions of investigation focus on special configurations of points. The study of such configurations, from a yet slightly different point of view, has been initiated by Bocci and Chiantini in [4]. We follow their approach in the present note.

For a homogeneous ideal $I = \bigoplus_{d \geq 0} I_d$, we define the *initial degree* $\alpha(I)$ of I as the least integer d such that $I_d \neq 0$. More generally, we define the *initial sequence* (or simply the α -sequence of I) as the strictly increasing sequence of integers

$$\alpha(I) < \alpha(I^{(2)}) < \alpha(I^{(3)}) < \alpha(I^{(4)}) < \dots, \quad (1)$$

where $I^{(m)}$ denotes the m -th symbolic power of I .

There is a related asymptotic quantity introduced by Chudnovsky [7] and rediscovered by Harbourne who named it the *Waldschmidt constant* of I :

$$\widehat{\alpha}(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m} = \inf_{m \geq 1} \frac{\alpha(I^{(m)})}{m}.$$

For radical ideals $I = I(Z)$ attached to configurations Z of points in \mathbb{P}^2 Bocci and Chiantini studied the question to what extent the value of the difference $\alpha(I^{(2)}) -$

$\alpha(I)$ determines the geometry of Z . Their results, still for planar points configurations, have been considerably generalized in [12] and further extended to other varieties in [19], [1], [3] and [10]. Many ideas presented here can be adapted to a more general setting. We don't dwell on this point in order to keep the presentation as transparent as possible.

It is convenient to define the *first differences sequence* of the initial sequence as $\beta_m(I) = \alpha(I^{(m+1)}) - \alpha(I^{(m)})$ for $m \geq 1$ and to set $\beta_0(I) = \alpha(I)$. We call this sequence simply the β -sequence of I . Of course, the α sequence determines the β sequence and vice versa. In the present note we focus on zero sets of ideals whose β -sequence contains many twos and threes.

Our main result is the following classification statement, see Definition 2.2 for the terminology applied.

Main Theorem. *Let $Z = \{P_1, \dots, P_s\}$ be a finite set of points in \mathbb{P}^2 and let $I = I(Z)$ be its radical ideal. If*

$$\widehat{\alpha}(I) < \frac{9}{4},$$

then Z

- a) *is contained in a line ($\widehat{\alpha}(I) = 1$) or a conic ($\widehat{\alpha}(I) \leq 2$) or;*
- b) *is a 4-star ($\widehat{\alpha}(Z) = 2$) and $s = 6$ in this case.*

Moreover if $\alpha(I^{(m)}) = 9/4$ for some m , then Z is a 3-quasi star.

As sample consequences of the Main Theorem we derive the following results.

Corollary A. *Let I be the radical ideal of a finite set Z of points in \mathbb{P}^2 .*

- a) *If there exists $m \geq 1$ such that $\beta_m(I) = \beta_{m+1}(I) = 1$, then $\alpha(I) = 1$, i.e. the set Z is contained in a line.*
- b) *If there exists $m \geq 1$ such that $\beta_m(I) = \beta_{m+1}(I) = \beta_{m+2}(I) = \beta_{m+3}(I) = \beta_{m+4}(I) = 2$, then $\alpha(I) = 2$ i.e. Z is contained in a conic.*
- c) *For any $d \geq 3$ there exist configurations of points such that $\beta_m(I) = d$ for all $m \geq 2$ but $\alpha(I) \geq d + 1$.*

Corollary B. *Let I be the radical ideal of a finite set Z of points in \mathbb{P}^2 with an integral Waldschmidt constant $\widehat{\alpha}(I) = d$.*

- (i) *if $d = 1$, then $\alpha(I) = 1$, i.e. Z is contained in a line;*
- (ii) *if $d \geq 2$, then $\alpha(I)$ need not to be equal d , i.e. there exist configurations of points Z with $\alpha(I) \geq d + 1$.*

We work over an algebraically closed field of characteristic 0.

2 Preliminaries and auxiliary results

In this section we fix the notation and recall very useful results of Chudnovsky and Esnault and Viehweg.

For a point $P \in \mathbb{P}^2$ let $I(P)$ denote the radical ideal containing all forms vanishing at P .

Let $Z = \{P_1, \dots, P_s\}$ be a fixed finite set of points in the projective plane. Then the ideal of Z is

$$I = I(Z) = I(P_1) \cap \cdots \cap I(P_s).$$

In this situation, for a positive integer m , the m -th symbolic power of I is defined by

$$I^{(m)} := I(P_1)^m \cap \cdots \cap I(P_s)^m, \quad (2)$$

see [21, Chapter IV.12, Definition] for the general definition. It is convenient and customarily to denote by mZ the subscheme of \mathbb{P}^2 defined by the ideal $I^{(m)}$. We will also write $\alpha(mZ)$ rather than $\alpha(I^{(m)})$ if we are primarily interested in the geometry of the set Z .

The values of the initial sequence (1) were considered already by Chudnovsky in [7] with the notation $\alpha(mZ) = \Omega(Z, m)$. He showed [7, General Theorem 6] that one has always the following inequality (for sets Z of points in \mathbb{P}^2):

$$\frac{\alpha(Z) + 1}{2} \leq \frac{\alpha(mZ)}{m}.$$

This result has been generalized by Esnault and Viehweg [14, Inequality (A), page 76]. For any $n \leq m$ we have

$$\frac{\alpha(nZ) + 1}{n+1} \leq \frac{\alpha(mZ)}{m}. \quad (3)$$

As the immediate corollary we have

$$\frac{\alpha(mZ) + 1}{m+1} \leq \hat{\alpha}(Z) \leq \frac{\alpha(mZ)}{m} \quad (4)$$

for any $m \geq 1$. We have also the following useful reformulation of (3).

Proposition 2.1. *Let I be the radical ideal of a finite set of points Z in the projective plane. Let $d \geq k \geq 2$ and $m \geq 1$ be fixed integers such that*

$$\alpha((m+k)Z) = \alpha(mZ) + d. \quad (5)$$

Then

$$\alpha((m+k)Z) \leq \frac{d-1}{k-1}(m+k). \quad (6)$$

Proof. Plugging (5) into (3) we get

$$\frac{\alpha((m+k)Z) - (d-1)}{m+1} \leq \frac{\alpha((m+k)Z)}{m+k}.$$

Resolving this inequality with respect to $\alpha((m+k)Z)$ yields (6). \square

In the sequel we will encounter some interesting geometrical configurations of points.

Definition 2.2 (Star configuration of points). We say that $Z \subset \mathbb{P}^2$ is a *star configuration* of degree d (or a d -star for short) if Z consists of **all** intersection points of d **general** lines in \mathbb{P}^2 .

Star configurations can be defined much more generally and they pop up frequently in situation similar to those studied here. We refer to [16] for a very nice introduction to this circle of ideas.

We will need also the following modification of Definition 2.2.

Definition 2.3 (Quasi star configuration of points). We say that $Z \subset \mathbb{P}^2$ is a *quasi star configuration* of degree d (or a d -quasi star for short) if Z consists of all intersection points of d general lines in \mathbb{P}^2 and additionally there is exactly one more point from Z on each of the lines, moreover these additional points are not collinear.

Note that if the extra points in the above definition were collinear Z would be a $(d+1)$ -star. Note also that a d -star contains exactly $d(d-1)/2$ points and a d -quasi star contains exactly $d(d+1)/2$ points. The figure below depicts a 3-quasi star.

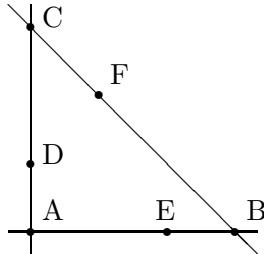


Figure 1

In the sequel we use without further comments the convention that the line passing through the points P_i and P_j is denoted by L_{ij} and the line through P and Q by L_{PQ} .

2.1 Bezout decomposition

We conclude this section with the following useful concept derived from the Bezout's Theorem, see [15, Proposition 8.4]. Let $Z = \{P_1, \dots, P_s\}$ be a finite set of points in the projective plane \mathbb{P}^2 . Let D be an effective divisor of degree d vanishing to order m_1, \dots, m_s at the points P_1, \dots, P_s respectively. Let C_1, \dots, C_r be irreducible curves of degrees c_1, \dots, c_r respectively and with $m_j^i = \text{mult}_{P_j} C_i$ for $i = 1, \dots, r$ and $j = 1, \dots, s$. Then we run the following algorithm. For each i we compare the two numbers

$$d_i := D \cdot C_i = dc_i \quad \text{and} \quad e_i := \sum_{j=1}^s m_j m_j^i.$$

The first number is of course the intersection number of the divisor D and the curve C_i . The second number appears in the Bezout's Theorem, which asserts that

- either $d_i \geq e_i$ holds;
- or C_i is a component of D .

We decompose D as

$$D = \sum_{i: e_i > d_i} C_i + D'$$

and repeat the procedure for the divisor D' . After a finite number of steps we obtain the following decomposition

$$D = \sum_{i=1}^r a_i C_i + B(D) \tag{7}$$

with $a_i \geq 0$ for $i = 1, \dots, r$ and the divisor $B(D)$ satisfying

$$B(D) \cdot C_i \geq \sum_{j=1}^s \left(m_j - \sum_{k=1}^r a_k m_j^k \right) m_j^i \quad (8)$$

for all $i = 1, \dots, r$.

Definition 2.4 (Bezout decomposition). We call the decomposition in (7) the *Bezout decomposition* of D with respect to the set Z and curves C_1, \dots, C_r and the divisor $B(D)$ the *Bezout reduction* of D .

Remark 2.5. Note that the Bezout decomposition is determined purely numerically. That implies in particular that if C_i and $C_{i'}$ are irreducible curves with the same degree and the same multiplicities in points from the set Z , then they will appear in (7) with the same coefficients a_i and $a_{i'}$.

Theorem 2.6 (Uniqueness of the Bezout decomposition). *The Bezout decomposition defined in 2.4 is uniquely determined.*

Proof. Keeping the notation introduced in this paragraph let $f : X \rightarrow \mathbb{P}^2$ be the blow up of the points P_1, \dots, P_s with exceptional divisors E_1, \dots, E_s . We denote by $\tilde{\Gamma} = f^*\Gamma - \sum_{i=1}^s \text{mult}_{P_i} \Gamma \cdot E_i$ the proper transform on X of a curve $\Gamma \subset \mathbb{P}^2$. The condition $e_i > d_i$ is equivalent to $\tilde{D} \cdot \tilde{C}_i < 0$. This immediately implies that \tilde{C}_i is a component of \tilde{D} . Thus the Bezout decomposition of D on \mathbb{P}^2 corresponds to subtracting from \tilde{D} those curves among $\tilde{C}_1, \dots, \tilde{C}_r$ which have negative intersection with \tilde{D} (this can be viewed as a first reduction in taking the Zariski decomposition of \tilde{D}). It suffices now to show that this reduction does not depend on the order in which the curves are subtracted.

This is a consequence of the following simple observation. Suppose that $\tilde{D} \cdot \tilde{C} < 0$ and let \tilde{D}' be a divisor obtained from \tilde{D} by subtracting a curve $\tilde{\Gamma}$ different from \tilde{C} . Then we have still $\tilde{D}' \cdot \tilde{C} < 0$ and have to subtract the curve \tilde{C} from \tilde{D}' according to our algorithm. This shows that *locally* the change of order in the reduction procedure does not influence the resulting divisor. This means in turn that the Bezout decomposition is *locally* confluent. Since the algorithm has to stop after finitely many steps, it is also *globally* confluent and the uniqueness of the Bezout reduction divisor $B(D)$ follows easily from an elementary version of the Church-Rosser Theorem [8]. \square

3 Configurations of points with $\widehat{\alpha}(Z) < \frac{9}{4}$

In this section we will prove the Main Theorem. Of course if Z is contained in a line or in a conic then $\widehat{\alpha}(Z) = 1$ or $\widehat{\alpha}(Z) \leq 2$ respectively (see Proposition 3.3 for the complete list of sets Z with Waldschmidt constants < 2). So we restrict our attention to sets not contained in a conic. Thus let $Z = \{P_1, \dots, P_s\}$ be a finite set of points in \mathbb{P}^2 not contained in a conic, so that in particular $s \geq 6$ holds. We can assume, renumbering the points if necessary, that the subset $W = \{P_1, \dots, P_6\}$ is not contained in a conic. In order to complete the proof of the Main Theorem we need to show that $Z = W$ and Z is a 4-star. The proof splits into several cases.

Note to begin with that the assumption $\widehat{\alpha}(Z) < \frac{9}{4}$ implies that there exists $m \geq 1$ such that

$$\frac{\alpha(mZ)}{m} \leq \frac{9}{4} \quad (9)$$

We fix such m and write it in the form

$$m = 4n + p \text{ with } 0 \leq p \leq 3.$$

Then

$$\alpha(mZ) \leq 9n + 2p.$$

By (9) there exists a divisor Γ of degree $9n + 2p$ vanishing along $(4n + p)W$. We work with this divisor throughout the proof. We split the proof in a number of cases.

Case 1. Assume that no three points in W are collinear. Then each conic, passing through exactly five points of W , is irreducible. We denote by C_i the conic passing through all points in W but P_i for $i = 1, \dots, 6$ and perform the Bezout decomposition of Γ with respect to W and the conics C_1, \dots, C_6 . Since the situation is symmetric with respect to these curves, we end up with the following decomposition

$$\Gamma = k(C_1 + \dots + C_6) + B(\Gamma)$$

with the inequality

$$2(9n + 2p - 12k) \geq 5(4n + p - 5k)$$

coming from (8). Equivalently we have

$$k \geq 2n + p. \quad (10)$$

On the other hand the degree of the Bezout reduction divisor $B(\Gamma)$ must satisfy

$$9n + 2p - 12k \geq 0. \quad (11)$$

But (10) and (11) are contradictory for $m > 0$. Hence there are at least 3 collinear points in W .

Case 2. Three points in W are collinear. Without loss of generality let P_1, P_2, P_3 lie on a line L . We keep this assumption until the end of the proof.

Observe that no more points of W lie on L , since otherwise W would lie on a conic. Similarly, P_4, P_5, P_6 cannot be collinear hence they determine three distinct lines L_{45}, L_{46} and L_{56} . The union of these lines (the triangle determined by the set $\{P_4, P_5, P_6\}$) is denoted by $T = L_{45} + L_{46} + L_{56}$. T may or may not contain some of the points P_1, P_2, P_3 . The rest of the proof splits onto subcases depending on how many of the points P_1, P_2, P_3 lie on T .

Subcase 2.0. The triangle does not pass through any of the points P_1, P_2, P_3 . This situation is depicted below.

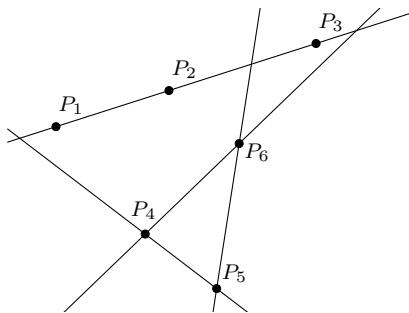


Figure 2

We consider now the Bezout decomposition of Γ with respect to the set W and the lines L , L_{45} , L_{46} and L_{56} . By symmetry the Bezout decomposition of Γ has the shape

$$\Gamma = kL + \ell T + B(\Gamma).$$

It is easy to check that $k \geq 1$ in this case. Note that the degree of $B(\Gamma)$ is $9n + 2p - k - 3\ell$ and this divisor vanishes along $(4n + p - k)X + (4n + p - 2\ell)Y$, where $X = \{P_1, P_2, P_3\}$ and $Y = \{P_4, P_5, P_6\}$. We have the following three inequalities:

$$9n + 2p - k - 3\ell \geq 3(4n + p - k) \quad (\text{the intersection } B(\Gamma) \cdot L) \quad (12)$$

$$9n + 2p - k - 3\ell \geq 2(4n + p - 2\ell) \quad (\text{the intersection of } B(\Gamma) \text{ with a line in } T) \quad (13)$$

$$9n + 2p - k - 3\ell \geq 0 \quad (\text{the nonnegativity of the degree of } B(\Gamma)). \quad (14)$$

The inequality (13) gives $n \geq k - \ell$ and together with (12) we obtain

$$2k - 3\ell - p \geq 3n \geq 3k - 3\ell,$$

which implies $k + p \leq 0$, a contradiction.

Subcase 2.1 The triangle T contains exactly one of the points $\{P_1, P_2, P_3\}$. Without loss of generality we may assume that it is the point P_1 and that it lies on the line L_{45} . This situation is depicted in the figure below.

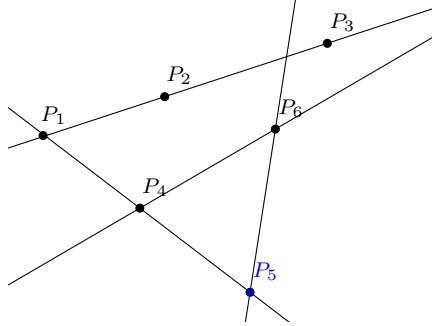


Figure 3

Now, the situation is symmetric with respect to the lines L (through P_1, P_2 and P_3) and L_{45} (which passes also through the point P_1). It is also symmetric with respect to the lines L_{46}, L_{56} and L_{26}, L_{36} (which are not indicated in the above picture). Performing the Bezout decomposition of Γ with respect to these lines now, we obtain

$$\Gamma = k(L + L_{45}) + \ell(L_{46} + L_{56} + L_{26} + L_{36}) + B(\Gamma).$$

The curve $B(\Gamma)$ has degree $9n + 2p - 2k - 4\ell$ and it vanishes to order $4n + p - 2k$ at P_1 , $4n + p - 4\ell$ at P_6 and $4n + p - k - \ell$ at all other points of W . Hence we have the following inequalities

$$9n + 2p - 2k - 4\ell \geq 2(4n + p - k - \ell) + (4n + p - 2k) \quad (15)$$

$$9n + 2p - 2k - 4\ell \geq (4n + p - k - \ell) + (4n + p - 4\ell) \quad (16)$$

$$9n + 2p - 2k - 4\ell \geq 0. \quad (17)$$

The second inequality gives $n \geq k - \ell$ whereas the first one implies $2(k - \ell) \geq 3n + p$. Hence $2n \geq 2(k - \ell) \geq 3n + p$, which is absurd.

Subcase 2.2 Two points out of $\{P_1, P_2, P_3\}$ lie on T . Without loss of generality we may assume that P_1 lies on the line L_{45} , and P_2 lies on L_{46} . This configuration is indicated in the figure below.

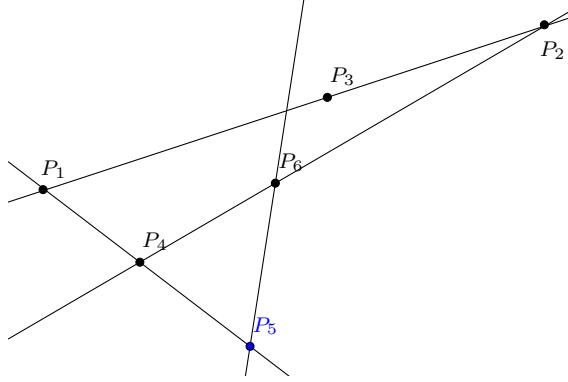


Figure 4

Let now $X = \{P_1, P_2, P_4\}$ and $Y = \{P_3, P_5, P_6\}$. Note that the set Y lies on the triangle T_X defined by X , each point on exactly one line. However, no point from X lies on the triangle T_Y defined by the set Y . If $Z = W$, then Z is a 3-quasi star. We show in Proposition 3.1 that $\hat{\alpha}(Z) = 9/4$ in this case.

Thus we may assume that there exists an extra point $P_7 \in Z$. Applying the Bezout decomposition with respect to the lines in triangles T_X and T_Y we get

$$\Gamma = kT_X + \ell T_Y + B(\Gamma)$$

with $B(\Gamma)$ vanishing to order $4n + p - 2k$ along X and to order $4n + p - k - 2\ell$ along Y . Thus we obtain the following inequalities:

$$9n + 2p - 3k - 3\ell \geq 2(4n + p - 2k) + (4n + p - k - 2\ell) \quad (18)$$

$$9n + 2p - 3k - 3\ell \geq 2(4n + p - k - 2\ell) \quad (19)$$

We observe additionally that removing each triangle T_X or T_Y from Γ causes the multiplicity of the residual divisor at the point P_7 to drop at most by one. So comparing the degree of the divisor $B(\Gamma)$ and its multiplicity at P_7 we obtain the following inequality

$$9n + 2p - 3k - 3\ell \geq 4n + p - k - \ell \quad (20)$$

From (18) we get $2k - \ell \geq 3m + p$, and from (19) we get $\ell - k \geq -n$. Beginning with the inequality (20) (after the reduction of terms) we obtain

$$5n \geq 2k + 2\ell - p = 6(\ell - k) + 4(2k - \ell) - p \geq -6n + 4(3n + p) - p = 6n + 3p$$

which is absurd.

Subcase 2.3 The set $\{P_1, P_2, P_3\}$ lies on a triangle defined by $\{P_4, P_5, P_6\}$. Then W is a 4-star. If $Z = W$ we are done. Otherwise consider an extra point $P_7 \in Z$. We denote by Δ the union of the 4 lines determined by W . Our two final cases depend on whether P_7 lies on Δ or not.

Subsubcase 2.3.a We assume first that the point P_7 does not lie on Δ . The situation is indicated in the figure below.

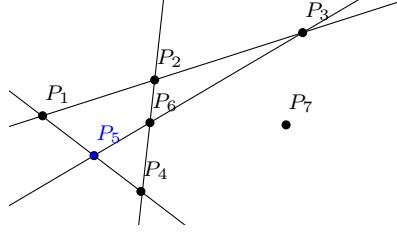


Figure 5

Let

$$\Gamma = k\Delta + B(\Gamma)$$

be the Bezout decomposition of Γ with respect to the lines in Δ , so that

$$9n + 2p - 4k \geq 3(4n + p - 2k) \quad (21)$$

holds. Comparing the degree $9m + 2p - 4k$ of the divisor $B(\Gamma)$ with its multiplicity $4n + p$ at P_7 we obtain additionally that

$$9n + 2p - 4k \geq 4n + p \quad (22)$$

Reducing terms in inequalities (21) and (22) we get

$$2k \geq 3n + p \quad \text{and} \quad 5n \geq 4k - p,$$

which gives a contradiction.

Subsubcase 2.3.b Now we pass to the final case and assume that P_7 is contained in Δ . Without loss of generality we assume that P_7 lies on the line L defined by P_1 and P_2 . This is indicated by the following figure.

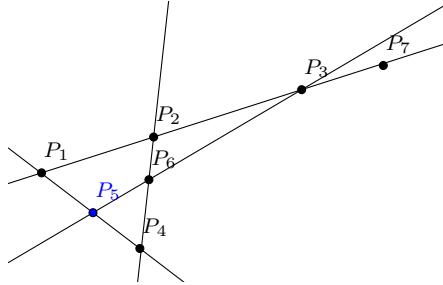


Figure 6

Let now $X = \{P_1, P_2, P_3\}$ and $Y = \{P_4, P_5, P_6\}$. For the Bezout decomposition we consider now the following divisors: L , the triangle T_Y and the pencil $\Pi = L_{74} + L_{75} + L_{76}$. Let

$$\Gamma = kL + \ell T_Y + t\Pi + B(\Gamma)$$

be the Bezout decomposition. The divisor $B(\Gamma)$ has degree $9n + 2p - k - 3\ell - 3t$ and vanishes along the points in X to order $4n + p - k - \ell$, along Y to order $4n + p - 2\ell - t$ and to order $4n + p - k - 3t$ at P_7 . So we have the following system of inequalities:

$$9n + 2p - k - 3\ell - 3t \geq 3(4n + p - k - \ell) + (4n + p - k - 3t) \quad (23)$$

$$9n + 2p - k - 3\ell - 3t \geq (4n + p - k - \ell) + 2(4n + p - 2\ell - t) \quad (24)$$

$$9n + 2p - k - 3\ell - 3t \geq (4n + p - 2\ell - t) + (4n + p - k - 3t) \quad (25)$$

$$9n + 2p - k - 3\ell - 3t \geq 0 \quad (26)$$

After reductions, the first three inequalities (23)–(25) give the following simpler system of inequalities:

$$\begin{aligned} 3k &\geq 7n + p, \\ 2\ell - t &\geq 3n + p, \\ t - \ell &\geq -n. \end{aligned}$$

Hence we have

$$k \geq \frac{1}{3}(7n + 2p), \quad \ell \geq 2n + p \text{ and } t \geq n + p,$$

which combined with (26) gives

$$0 \leq 9n + 2p - k - 3\ell - 3t \leq 9n + 2p - \frac{1}{3}(7n + 2p) - 3(2n + p) - 3(n + p) \leq -\frac{7}{3}n - \frac{14}{3}p < 0.$$

Thus we are done with claims a) and b) of the Main Theorem.

The "moreover" part follows from Case 2.2 above and the contradictions in all other cases. \square

Now we will compute the Waldschmidt constant of the 3-quasi star.

Proposition 3.1 (A 3-quasi star). *Let Z be a 3-quasi star. Then $\widehat{\alpha}(Z) = \frac{9}{4}$.*

Proof. We use the notation as in the Figure 1, thus let $Z = \{A, B, C, D, E, F\}$. Note to begin with that taking

$$\Delta = 2(L_{AB} + L_{BC} + L_{AC}) + L_{DE} + L_{EF} + L_{DF}$$

we obtain a divisor of degree 9 vanishing to order 4 along Z . This shows that

$$\widehat{\alpha}(Z) \leq \frac{9}{4}.$$

In order to prove the reverse inequality, assume that there exists a divisor Γ of degree d vanishing along Z to order m and such that

$$\frac{d}{m} < \frac{9}{4}. \tag{27}$$

We may also assume that Γ has the least degree d such that (27) holds. It is easy to check, using the Bezout's Theorem, that the divisor Δ defined above has to be contained in Γ . But then, for $\Gamma' = \Gamma - \Delta$ we have $d' = d - 9$ and $m' = m - 4$. Hence $d'/m' < 9/4$ holds as well, and this contradicts the minimality of d . \square

Remark 3.2. We don't know if the above Proposition can be reversed, i.e. if a 3-quasi star is the only configuration with the Waldschmidt constant equal $9/4$. There could exist a set $Z \subset \mathbb{P}^2$ such that $\alpha(mZ) > 9/4$ for all $m \geq 1$ but $\widehat{\alpha}(Z) = 9/4$.

We conclude this section with the classification of all point configurations with Waldschmidt constants less than 2.

Proposition 3.3 (Waldschmidt constants < 2). *Let Z be a finite set of points in \mathbb{P}^2 with $\widehat{\alpha}(Z) < 2$. Then*

$$\widehat{\alpha}(Z) = \frac{2k - 1}{k} \quad \text{for some } k \in \mathbb{Z}_{>0}$$

and Z consists of k points P_1, \dots, P_k contained in a line L and a single point Q not contained in L .

Proof. Suppose that Z is not of the form asserted in the Proposition. Then there exist points $P, Q, R, S \in Z$ such that no 3 of them are collinear. There exists a divisor Γ of degree d vanishing to order m along Z , so in particular at points P, Q, R, S , with $d < 2m$. Let M be the line through P and Q and let N be the line determined by R and S . Let

$$\Gamma = k(M + N) + B(\Gamma)$$

be the Bezout decomposition of Γ with respect to the lines M and N . Then $\deg(B(\Gamma)) = d - 2k$ and

$$d - 2k \geq 2(m - k)$$

holds. But this is equivalent to $d \geq 2m > d$. A contradiction.

Now we calculate $\widehat{\alpha}(Z)$ for sets Z described in the Proposition. Let L_i be the line through Q and P_i . For the divisor

$$\Gamma = (k-1)L + L_1 + \dots + L_k$$

we have $\deg(\Gamma) = 2k - 1$ and $\text{mult}_P(\Gamma) = k$ for all $P \in Z$. Hence $\widehat{\alpha}(Z) \leq \frac{2k-1}{k}$.

It remains to show the Waldschmidt constant is exactly $\frac{2k-1}{k}$. To this end assume that there exists a divisor Γ vanishing to order m at all points in Z and of degree d satisfying $d < \frac{2k-1}{k}m$. Let $\Pi = L_1 + \dots + L_k$. Let

$$\Gamma = pL + q\Pi + B(\Gamma)$$

be the Bezout decomposition of Γ with respect to the lines L and L_1, \dots, L_k . Then we have the following inequalities:

$$\begin{aligned} d - p - kq &\geq k(m - p - q) \\ d - p - kq &\geq m - p - q + m - kq. \end{aligned}$$

Of course $\deg(B(\Gamma)) = d - p - kq \geq 0$ holds. But all these three inequalities are contradictory. \square

4 Configurations with low β -sequences

In this section we prove Corollaries announced in the Introduction.

Proof of Corollary A. Part a) was already proved as Corollary 3.5 in [12].

Part b) with $m = 1$ was also already proved as Theorem 4.11 in [12] and the general case was conjectured there [12, Conjecture 4.6]. The argument presented here is new and covers both cases.

We have by assumption that $\alpha((m+5)Z) = \alpha(mZ) + 10$, hence Proposition 2.1 with $k = 5$ and $d = 10$ gives

$$\alpha((m+5)Z) \leq \frac{9}{4}(m+5).$$

Thus the Main Theorem and Proposition 3.1 applies. Since the β -sequence for the 4-star is

$$3, 1, 3, 1, 3, 1, 3, 1, \dots$$

and for the 3-quasi star it is

$$3, 2, 2, 2, 3, 2, 2, 2, 3, 2, 2, 2, \dots$$

these cases are excluded and we conclude that Z is contained in a conic.

Part c) for $d \geq 4$ follows from [12, Example 4.14] and for $d = 3$ we construct a new example in the proof of Proposition 4.1 below. \square

Now we pass to the second corollary.

Proof of Corollary B. Part a) follows immediately from Chudnovsky inequality (4) with $m = 1$.

Part b) for $d \geq 3$ follows from [12, Example 4.14], whereas for $d = 2$ a 4-star provides an example. \square

Proposition 4.1. *There exists a configuration Z of points in \mathbb{P}^2 such that for the radical ideal $I = I(Z)$*

$$\alpha(I^{(m)}) = 3m \text{ for all } m \geq 2$$

but $\alpha(I) = 4$.

Proof. We provide an explicit example. To this end we consider the subscheme Z consisting of ten points P_1, \dots, P_{10} such that P_1, P_2, P_3 are the intersection points of three general lines L_1, L_2, L_3 , points P_4, \dots, P_9 lie in pairs in general position on lines L_1, L_2, L_3 and P_{10} is a general point in \mathbb{P}^2 . This is indicated in the figure below.

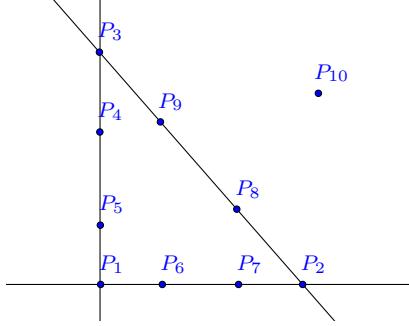


Figure 7

It is easy to see that $\alpha(Z) = 4$ in this situation. We claim that for $k \geq 2$, $\alpha(kZ) = 3k$. Indeed, $\alpha(kZ) \leq 3k$ as for every $k \geq 2$ there exists a divisor of degree $3k$ passing through Z with multiplicities k composed of:

1. $t(L_1 + L_2 + L_3) + tC$, if $k = 2t$
2. $t(L_1 + L_2 + L_3) + (t - 1)C + S$, if $k = 2t + 1$

where C is a cubic with a double point in P_{10} passing through P_4, \dots, P_9 , while S is the sextic passing through P_1, P_2, P_3 , having multiplicity 2 at P_4, \dots, P_9 and multiplicity 3 at P_{10} .

It suffices to prove that there is no divisor of degree $3k - 1$ passing through Z with multiplicities k . Suppose that such a divisor D exists. Let $D = a(L_1 + L_2 + L_3) + bC + B(D)$ be the Bezout decomposition of D with respect to the curves L_1, L_2, L_3 and C . As the degree of $B(D)$ is $3k - 3a - 3b - 1$ and $B(D)$ passes through Z with multiplicities

$$k - 2a \text{ at } P_1, P_2, P_3; k - a - b \text{ at } P_4, \dots, P_9 \text{ and } k - 2b \text{ at } P_{10},$$

we obtain

$$\begin{aligned} 3k - 3a - 3b - 1 &\geq 2(k - 2a) + 2(k - a - b) \\ 3(3k - 3a - 3b - 1) &\geq 2(k - 2b) + 6(k - a - b). \end{aligned}$$

Adding these inequalities and reducing terms we get $-4 \geq 0$, a contradiction. \square

Remark 4.2. In the example above the β -sequence is $2, 3, 3, 3, \dots$ with $\beta_0 = 4$. We don't know if there exists a set $Z \subset \mathbb{P}^2$ with the β -sequence constantly equal 3 and $\beta_0 = 4$. It would be desirable to know this in the view of [12, Example 4.14].

Motivated by the above Remark, we provide now an example of a subscheme Z such that there is a considerable number of threes in the beginning of its β -sequence.

Example 4.3. Let $\{P_1, \dots, P_7\}$ be generic points on an irreducible conic. Let P_8 be the intersection point of $L_{1,2}$ and $L_{6,7}$, let P_{10} be the intersection point of $L_{2,3}$ and $L_{4,5}$ and let P_9 be the intersection point of $L_{1,10}$ and $L_{6,7}$. This assumptions are illustrated in Figure 8.

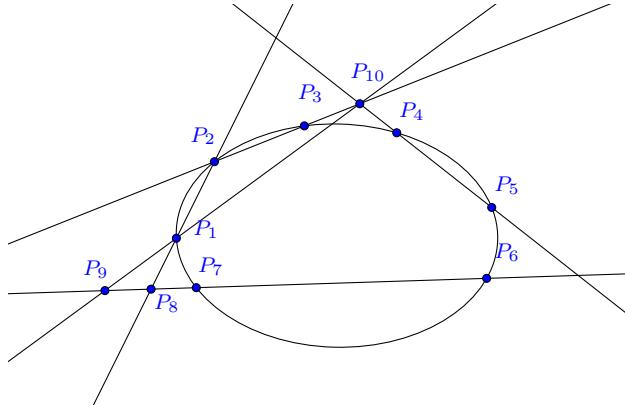


Figure 8

Then using any algebra computer program (the authors did it with Singular [9]) one can check that this configuration of $\{P_1, \dots, P_{10}\}$ indeed satisfies $\alpha(Z) = 4$, $\alpha(kZ) - \alpha((k-1)Z) = 3$ for $k = 2, \dots, 29$ and $\alpha(30Z) - \alpha(29Z) = 4$.

We conclude this note with the following.

Remark 4.4. It would be interesting to see to what extend the classification stated in the Main Theorem can be prolonged for higher values of $\hat{\alpha}(Z)$. We expect the problem to be feasible for all Z with $\hat{\alpha}(Z) < 3$. We hope to come back to this issue soon.

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